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$\mathcal{W}$-superalgebras as truncations of super-Yangians

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Abstract

We show that some finite $\mathcal{W}$-superalgebras based on $gl(M|N)$ are truncations of the super-Yangian $Y(gl(M|N))$. In the same way, we prove that finite $\mathcal{W}$-superalgebras based on $osp(M|2n)$ are truncations of the twisted super-Yangians $Y(gl(M|2n))^+$. Using this homomorphism, we present these $\mathcal{W}$-superalgebras in an R-matrix formalism, and we classify their finite-dimensional irreducible representations.

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1. Introduction

$\mathcal{W}$-algebras have been introduced in the 2d-conformal models as a tool for the study of these theories. Then, these algebras and their finite-dimensional versions appeared to be relevant in several physical backgrounds. For more details on $\mathcal{W}$-algebras, see e.g. [1]. However, a full understanding of their algebraic structure (and of their geometrical interpretation) is lacking. The connection of some of these finite $\mathcal{W}$-algebras with Yangians appeared to be a solution at least for the algebraic structure. It could be surprising that Yangians [2], which play an important role in integrable systems, see e.g. [3], enter into the study of algebras originating from 2d-conformal models. Let us however note that such a connection has already been remarked in WZW models [4]. For more information on the algebraic structure of Yangians, see e.g. [5] and references therein.

The existence of an algebra homomorphism between a Yangian based on $sl(N)$ and finite $\mathcal{W}(sl(Np), N.sl(p))$ algebras was first proved in [6]. Such a connection plays a role in the study of physical models: for instance, in the case of the N-vectorial non-linear Schrödinger equation on the real line, the full symmetry is the Yangian $Y(gl(N)) \equiv Y(N)$, but the space of states with particle number less than p is a representation of the $\mathcal{W}(gl(Np), p.sl(N))$ algebra [7].

[1] UMR 5108 du CNRS, associée à l'Université de Savoie.

Later, the connection between Yangians and finite $\mathcal{W}(gl(Np), N.sl(p))$ algebras was proved in the FRT presentation [9] of the Yangian. It appears that in this framework the above $\mathcal{W}$-algebras are nothing but truncations of the Yangian $Y(N)$, p indicating the level where the truncation occurs. Thanks to this presentation, an (evaluated) R-matrix for these $\mathcal{W}$-algebras was given, and their finite-dimensional irreducible representations were classified [8].

Then, this connection was extended to a class of $\mathcal{W}$-algebras, namely the algebras of type $\mathcal{W}[so(2mp), m.sl(p)]$, $\mathcal{W}[so((2m+1)p), m.sl(p)+so(p)]$ and $\mathcal{W}[sp(2np), n.sl(p)]$, which were related to truncations of twisted Yangian $Y^{\pm}(N)$[10]. Note that although Yangians based on orthogonal and symplectic algebras exist [2], and admit an FRT presentation [11], it is the twisted Yangians introduced by Olshanski [12, 13] which enter into the game. The latter are not Hopf algebras but only Hopf co-ideals in $Y(N)$. Nevertheless, this relation allows us to give an R-matrix presentation of the $\mathcal{W}$-algebras under consideration, with however the slight change that it is an 'RSRS' relation which occurs,

$$R_{12}(u-v)S_1(u)R'_{12}(u+v)S_2(v) = S_2(v)R'_{12}(u+v)S_1(u)R_{12}(u-v)$$

differing from the usual FRT (also called 'RTT') presentation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v).$$

The classification of finite-dimensional irreducible representations of the $\mathcal{W}$-algebras then follows [10].

The aim of the present paper is to extend the above correspondence to the case of finite $\mathcal{W}$-superalgebras, based on Lie superalgebras $gl(M|N)$ and $osp(M|2n)$. As for $gl(N)$ on the one hand, and $so(m)$ and $sp(2n)$ on the other hand, the treatment for $gl(M|N)$ and for $osp(M|2n)$ will be very different. Due to this difference, this paper is divided into two main parts. In the first part, we show that $\mathcal{W}(gl(Mp|Np), (M+N)gl(p))$ superalgebras are truncations of the super-Yangian based on $gl(M|N)$, leading to an 'RTT' presentation of these $\mathcal{W}$-superalgebras. We use this property to classify the finite-dimensional irreducible representations of these $\mathcal{W}$-superalgebras. In the second part, we deal with $\mathcal{W}$-superalgebras based on $osp(M|N)$ and twisted super-Yangians. We show that these $\mathcal{W}$-superalgebras are truncations of twisted super-Yangians, leading to an 'RSRS' presentation of the former and a classification of their finite-dimensional irreducible representations.

2. Super-Yangian

The super-Yangian $Y(gl(M|N)) = Y(M|N)$ was first defined by Nazarov [14]. It can be obtained as the generalization of the construction for the Yangian $Y(M)$, based on the Lie algebra $gl(M)$, to the case of the Lie superalgebra $gl(M|N)$. Its representations have been studied by Zhang [15].

2.1. Introduction to $Y(M|N)$

The Lie superalgebra $gl(M|N)$ is a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$ spanned by the basis $\{\mathcal{E}_{ab}|a, b = 1, 2, \ldots, M+N\}$. We introduce the gradation index [ ]:

$$[a] = \begin{cases} 0 & \text{if} \quad a \leqslant M \\ 1 & \text{if} \quad M < a \leqslant M+N \end{cases} \qquad \text{and} \qquad [\mathcal{E}_{ab}] = [a]+[b]. \tag{2.1}$$

The bilinear graded commutator associated with $gl(M|N)$ is defined as follows:

$$[,\} : \begin{cases} gl(M|N)\otimes gl(M|N) \to gl(M|N) \\ [\mathcal{E}_{ab}, \mathcal{E}_{cd}\} = \delta_{cb}\mathcal{E}_{ad} - (-1)^{([a]+[b])([c]+[d])}\delta_{ad}\mathcal{E}_{cb}. \end{cases} \tag{2.2}$$

The super-Yangian $Y(M|N)$ is a $\mathbb{Z}_2$-graded Hopf algebra generated by an infinite set of elements $T^{ab}_{(n)}$, $a, b = 1, 2, \ldots, M+N$ and $n \in \mathbb{Z}_{>0}$. The $T^{ab}_{(n)}$ are even if $[a]+[b] \equiv 0 \pmod 2$ and odd otherwise.

We introduce the generating function

$$T(u) = \sum_{a,b=1}^{M+N} T^{ab}(u) E_{ab} \qquad \text{and} \qquad T^{ab}(u) = \sum_{n=0}^{\infty} T^{ab}_{(n)} u^{-n} \tag{2.3}$$

with $T^{ab}_{(0)} = \delta^{ab}$, u a spectral parameter and E_{ab} the matrix with 1 at position (a, b) and 0 elsewhere.

The following R-matrix

$$R(u) = \mathbb{1} \otimes \mathbb{1} - \frac{P}{u} \qquad u \in \mathbb{C}$$

satisfies the graded Yang–Baxter equation. The permutation operator P is defined by

$$P_{12} = \sum_{i,j} (-1)^{[j]} E_{ij} \otimes E_{ji} \tag{2.4}$$

and the tensor product is chosen graded,

$$(E_{ij} \otimes E_{kl}) \cdot (E_{mn} \otimes E_{pq}) = (-1)^{([k]+[l])([m]+[n])} E_{ij} E_{mn} \otimes E_{kl} E_{pq}. \tag{2.5}$$

The defining relations in $Y(M|N)$ can be written as follows,

$$R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v) \tag{2.6}$$

with

$$T_1(u) = \sum_{a,b=1}^{M+N} T^{ab}(u) E_{ab} \otimes \mathbb{1} \quad \text{and} \quad T_2(v) = \sum_{a,b=1}^{M+N} T^{ab}(v) \mathbb{1} \otimes E_{ab}. \tag{2.7}$$

We can rewrite equation (2.6) as follows,

$$[T^{ab}(u), T^{cd}(v)\} = \frac{(-1)^{[c]([a]+[b])+[a][b]}}{u - v} (T^{cb}(u) T^{ad}(v) - T^{cb}(v) T^{ad}(u)) \tag{2.8}$$

or equivalently

$$\begin{aligned}\left[T^{ab}_{(m)}, T^{cd}_{(n)}\right\} &= \delta^{cb} T^{ad}_{(m+n-1)} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad} T^{cb}_{(m+n-1)} \\ &\quad + (-1)^{[c]([a]+[b])+[a][b]} \sum_{r=1}^{\min -1} \left\{T^{cb}_{(r)} T^{ad}_{(m+n-1-r)} - T^{cb}_{(m+n-1-r)} T^{ad}_{(r)}\right\}\end{aligned} \tag{2.9}$$

where min stands for $\min(m, n)$.

The Hopf structure is given by

$$\epsilon(T^{ab}(u)) = \delta^{ab} \qquad S(T^{ab}(u)) = (T^{-1}(u))^{ab} \tag{2.10}$$

$$\Delta(T^{ab}(u)) = \sum_{e=1}^{M+N} T^{ae}(u) \otimes T^{eb}(u). \tag{2.11}$$

The super-Yangian $Y(M|N)$ is a deformation of the enveloping algebra of a polynomial algebra (restricted to its positive modes) based on $gl(M|N)$, denoted by $\mathcal{U}(gl(M|N)[x])$. The parameter $\hbar$ can be recovered after rescaling the generators by an appropriate power of $\hbar : T^{ab}_{(n)} \to \hbar^{n-1} T^{ab}_{(n)}$.

2.2. Finite-dimensional irreducible representations of $Y(M|N)$

The finite-dimensional irreducible representations of $Y(M|N)$ have been studied in [15]. We recall here the main results, using a different basis for the positive roots (see [17] for details).

We introduce the subsets $\mathbb{N}_{M+N} = [1, M+N] \cap \mathbb{Z}_+$, $\mathbb{N}^2_{M+N} = \mathbb{N}_{M+N} \times \mathbb{N}_{M+N}$ and the integer $\ell = \left[\frac{N}{2}\right]$. The definition of the positive roots will be associated with the set

$$\Phi^+ = \left\{(a,b) \in \mathbb{N}^2_{M+N} \quad \text{with either} \quad \left| \begin{array}{l} 1 \leqslant a < b \leqslant M \\ M+1 \leqslant a < b \leqslant M+N \\ 1 \leqslant a \leqslant M \text{ and } M+\ell+1 \leqslant b \leqslant M+N \\ M+1 \leqslant a \leqslant M+\ell \text{ and } 1 \leqslant b \leqslant M \end{array} \right. \right\}. \tag{2.12}$$

Definition 2.1. *Let V be an irreducible $Y(M|N)$-module. A nonzero element $v_+^\Lambda \in V$ is called a highest weight vector if*

$$\begin{aligned} &T^{ab}_{(n)} v_+^\Lambda = 0 \qquad \forall (a,b) \in \Phi_+ \quad n > 0 \\ &T^{aa}_{(n)} v_+^\Lambda = \lambda_a^{(n)} v_+^\Lambda \qquad a = 1, \ldots, M+N \quad n > 0 \quad \lambda_a^{(n)} \in \mathbb{C}. \end{aligned} \tag{2.13}$$

An irreducible module is called a highest weight module if it admits a highest weight vector. We define

$$\Lambda(u) \equiv (\lambda_1(u), \lambda_2(u), \ldots, \lambda_{M+N}(u)) \tag{2.14}$$

with $\lambda_a(u) = 1 + \sum_{n>0} \lambda_a^n u^{-n}$ and call $\Lambda(u)$ a highest weight of V.

Theorem 2.2. *Every finite-dimensional irreducible $Y(M|N)$-module V contains a unique (up to scalar multiples) highest weight vector v_+^Λ.*

Corresponding to each $\Lambda(u)$ of the form (2.14), there exists a unique irreducible highest weight $Y(M|N)$-module $V(\Lambda)$ with highest weight $\Lambda(u)$.

Theorem 2.3. *The irreducible highest weight $Y(M|N)$-module $V(\Lambda)$ is finite dimensional if and only if its highest weight $\Lambda(u)$ satisfies the following conditions,*

$$\begin{aligned} &\frac{\lambda_a(u)}{\lambda_{a+1}(u)} = \frac{P_a(u+1)}{P_a(u)} \qquad 1 \leqslant a < N+M \quad a \neq M \\ &\frac{\lambda_M(u)}{\lambda_{M+1}(u)} = \frac{\tilde{P}_M(u)}{P_M(u)} \end{aligned} \tag{2.15}$$

where, m_a being the degree of P_a,

$$\begin{aligned} &P_a(u) = \prod_{i=1}^{m_a} \left(u - \gamma_a^{(i)}\right) \qquad 1 \leqslant a < N+M \quad \textit{and} \quad a \neq M \quad \gamma_a^{(i)} \in \mathbb{C} \\ &\tilde{P}_M(u) = \prod_{i=1}^{m_M} \left(1 - \frac{\tilde{r}^{(i)}}{u}\right) \qquad \textit{and} \qquad P_M(u) = \prod_{i=1}^{m_M} \left(1 - \frac{r^{(i)}}{u}\right) \qquad r^{(i)}, \tilde{r}^{(i)} \in \mathbb{C}. \end{aligned} \tag{2.16}$$

Among the finite-dimensional highest weight representations, there is a class of particular interest:

Definition 2.4 (Evaluation representations)**.** *An evaluation representation ev_{π_μ} is a morphism from the super-Yangian $Y(M|N)$ to a highest weight irreducible representation π_μ of $gl(M|N)$. The morphism is given by*

$$ev_{\pi_\mu}(T^{ab}(u)) = \delta^{ab} + \pi_\mu(\mathcal{E}^{ab}) u^{-1} \qquad \forall a, b \in \{1, \ldots, M+N\} \tag{2.17}$$

that is

$$ev_{\pi_\mu}\big(T^{ab}_{(0)}\big) = \delta^{ab} \qquad ev_{\pi_\mu}\big(T^{ab}_{(1)}\big) = \pi_\mu(\mathcal{E}^{ab}) \qquad ev_{\pi_\mu}\big(T^{ab}_{(r)}\big) = 0 \qquad \text{for} \quad r > 1 \tag{2.18}$$

where $\mathcal{E}^{ab}$ are the standard $gl(M|N)$ generators.

The highest weight $\mu(u) = (\mu_1(u), \ldots, \mu_{M+N}(u))$ of the representation ev_{π_μ} is given by

$$\mu_a(u) = 1 + \mu_a u^{-1} \qquad \forall a \in \{1, \ldots, M+N\} \tag{2.19}$$

where $\mu = (\mu_1, \ldots, \mu_{M+N})$ is the highest weight of π_μ.

Any finite-dimensional irreducible representation can be obtained through the tensor products[2] of such evaluation representations [15]:

Definition 2.5 (Tensor product of evaluation representations)**.** *Let $\{ev_{\pi_i}\}_{i=1,\ldots,s}$ be a set of evaluation representations. The tensor product of these s representations $ev_{\vec{\pi}} = ev_{\pi_1} \otimes \cdots \otimes ev_{\pi_s}$ is a morphism from $Y(M|N)$ to the tensor product of $gl(M|N)$ representations $\vec{\pi} = \otimes_i \pi_i$ given by*

$$ev_{\vec{\pi}}\big(T^{ab}_{(r)}\big) = \bigoplus_{r_1+r_2+\cdots+r_n=r} \Big(ev_{\pi_1}\Big(T^{ai_1}_{(r_1)}\Big) \otimes ev_{\pi_2}\Big(T^{i_1 i_2}_{(r_2)}\Big) \otimes \cdots \otimes ev_{\pi_s}\Big(T^{i_{s-1}b}_{(r_s)}\Big)\Big) \tag{2.20}$$

where there is an implicit summation on the indices $i_1, i_2, \ldots, i_{s-1} = 1, \ldots, M+N$.

It satisfies

$$ev_{\vec{\pi}}\left(T^{ab}_{(r)}\right) \neq 0 \qquad \text{if and only if} \quad r \leqslant s. \tag{2.21}$$

2.3. Truncated super-Yangians

We will proceed as in [8]: we introduce $\mathcal{T}_p \equiv \mathcal{U}\big(\big\{T^{ij}_{(n)}, n > p\big\}\big)$ and the left ideal $\mathcal{I}_p \equiv Y(M|N) \cdot \mathcal{T}_p$ generated by $\mathcal{T}_p$. We then define the coset (truncation of the super-Yangian at order p),

$$Y_p(M|N) \equiv Y(M|N)/\mathcal{I}_p. \tag{2.22}$$

Property 2.6. *The truncated super-Yangian $Y_p(M|N)$ is a superalgebra ($\forall p \in \mathbb{Z}_{>0}$).*

Proof. As in [8] the Lie superalgebra structure of $Y_p(M|N)$ can be proved by showing that $\mathcal{I}_p$ is a two-sided ideal. We first show that

$$[Y(M|N), \mathcal{T}_p] \subset Y(M|N) \cdot \mathcal{T}_p = \mathcal{I}_p. \tag{2.23}$$

Relation (2.9) shows that $\big[T^{ij}_m, T^{kl}_n\big]$ (for $n > p$) is the sum of two terms, the first being in $Y(M|N) \cdot \mathcal{T}_p$, the second belonging to $\mathcal{T}_p \cdot Y(M|N)$. Focusing on the latter, one rewrites it as

$$\begin{aligned}
&\sum_{r=0}^{\mu-1} \left(T^{il}_r T^{kj}_{m+n-1-r} + (-1)^{[i]([k]+[j])+[k][j]} \sum_{s=0}^{r-1} \left(T^{ij}_s T^{kl}_{m+n-2-s} - T^{ij}_{m+n-2-s} T^{kl}_s \right) \right) \\
&\qquad = \sum_{r=0}^{\mu-1} T^{il}_r T^{kj}_{m+n-1-r} + (-1)^{[i]([k]+[j])+[k][j]} \\
&\qquad\quad \times \sum_{s=0}^{\mu-2} (\mu - s - 1) \left(T^{ij}_s T^{kl}_{m+n-2-s} - T^{ij}_{m+n-2-s} T^{kl}_s \right)
\end{aligned} \tag{2.24}$$

[2] Note however that one sometimes has to make a quotient to get an irreducible representation from these tensor products.

where μ stands for $\min(m, n)$. In (2.24), the first sum belongs to $\mathcal{I}_p$, while the last sum belongs to $\mathcal{T}_p \cdot Y(M|N)$, with a summation which has one term less than the previous one; we can thus proceed recursively in a finite number of steps. The final result is an element of $Y(M|N) \cdot \mathcal{T}_p$. In the same way, one can show that

$$[Y(M|N), \mathcal{T}_p] \subset \mathcal{T}_p \cdot Y(M|N) \tag{2.25}$$

so that $\mathcal{I}_p = Y(M|N) \cdot \mathcal{T}_p = \mathcal{T}_p \cdot Y(M|N)$. □

Note that Δ is not a morphism of this superalgebra (for the structure induced by $Y(M|N)$), i.e. $Y_p(M|N)$ has no natural Hopf structure.

Finally, we remark that each $Y_p(M|N)$ is a deformation of a truncated polynomial algebra based on $gl(M|N)$. By truncated polynomial algebra we mean the quotient of a usual $gl(M|N)$ polynomial algebra (of generators $T^{ij}_{(n)}$) by the relations $T^{ij}_{(n)} = 0$ for $n > p$. The construction is the same as for the full super-Yangian.

2.4. Poisson super-Yangians

In the following we will deal with classical super-Yangians, where the commutator is replaced by a $\mathbb{Z}_2$-graded Poisson bracket (PB). It corresponds to the usual classical limit of quantum groups. One sets

$$\begin{aligned}
&L(u) = \sum_{a,b=1}^{M+N} (-1)^{[b]} T^{ab}(u) \otimes E_{ba} \\
&R_{12}(u) = \mathbb{1} \otimes \mathbb{1} + \hbar r_{12}(u) + o(\hbar) \qquad \text{with} \quad r_{12}(u) = \frac{P_{12}}{u} \\
&[\,,\,] = \hbar\{\,,\,\} + o(\hbar).
\end{aligned}$$

Relation (2.6) is then expanded as a series in $\hbar$. Since in a classical super-Yangian we have $T^{ab}_{(n)} T^{cd}_{(m)} = (-1)^{([a]+[b])([c]+[d])} T^{cd}_{(m)} T^{ab}_{(n)}$, we obtain

$$\{T^{ab}(u), T^{cd}(v)\} = \frac{1}{u-v} (-1)^{[c]([a]+[b])+[a][b]} (T^{cb}(u) T^{ad}(v) - T^{cb}(v) T^{ad}(u)) \tag{2.26}$$

which leads to

$$\begin{aligned}
\left\{T^{ab}_{(m)}, T^{cd}_{(n)}\right\} &= \delta_{cb} T^{ad}_{(m+n-1)} - (-1)^{([a]+[b])([c]+[d])} \delta_{ad} T^{cb}_{(m+n-1)} \\
&\quad + (-1)^{[c]([a]+[b])+[a][b]} \sum_{r=1}^{\min(m,n)-1} \left(T^{cb}_{(r)} T^{ad}_{(m+n-1-r)} - T^{cb}_{(m+n-1-r)} T^{ad}_{(r)}\right).
\end{aligned} \tag{2.27}$$

In classical super-Yangians, all the algebraic properties described above still apply.

3. $\mathcal{W}(gl(Mp|Np), (M+N)gl(p))$ superalgebras

For simplicity we note $\mathcal{W}_p(M|N) \equiv \mathcal{W}(gl(Mp|Np), (M+N)gl(p))$.

3.1. Definition of $\mathcal{W}(\mathcal{G}, \mathcal{H})$ superalgebras and Dirac brackets

$\mathcal{W}(\mathcal{G}, \mathcal{H})$ (super)algebras can be constructed as Hamiltonian reduction on a Lie (super)algebra $\mathcal{G}$, with Poisson brackets $\{.\,,.\}$. The construction is done as follows.

We start with an $sl(2)$ embedding in $\mathcal{G}$, this embedding being defined as the principal embedding in a regular sub(super)algebra $\mathcal{H} \subset \mathcal{G}$. We recall that the principal $sl(2)$ embedding of an algebra $\mathcal{H}$ is given by $e_+ = \sum_i e_i$, where e_+ is the positive root generator of $sl(2)$, and

e_i are the simple root generators of $\mathcal{H}$. If $\mathcal{H}$ is a superalgebra, the principal $sl(2)$ embedding is defined as the principal embedding of its bosonic part.

Once the $sl(2)$ embedding in $\mathcal{G}$ is fixed (i.e. when $\mathcal{H} \subset \mathcal{G}$ is given), let $(e_\pm, h)$ be its generators, one decomposes $\mathcal{G}$ into $sl(2)$ representations. This amounts to taking a $\mathcal{G}$-basis of the form J^i_{jm}, $-j \leqslant m \leqslant j$, and i labelling the multiplicities, with

$$\left[e_\pm, J^i_{jm}\right] = \alpha_{jm} J^i_{j,m\pm1} \qquad \left[h, J^i_{jm}\right] = m J^i_{jm} \qquad \text{with} \quad \alpha_{j,m} \in \mathbb{C}. \tag{3.1}$$

We take $e_\pm = J^0_{1,\pm1}$ and $h = J^0_{1,0}$. Then, one introduces a set of second-class constraints (in Dirac terminology):

$$J^i_{jm} = \delta^{i,0}\delta_{j,1}\delta_{m,-1} \qquad \text{for} \quad m < j, \forall j, \forall i. \tag{3.2}$$

This amounts to setting to zero all the generators but the $sl(2)$ highest weight ones (which are left free), and e_- which is set to 1.

The $\mathcal{W}(\mathcal{G}, \mathcal{H})$ (super)algebra is defined as the enveloping algebra generated by the $sl(2)$ highest weight generators, equipped with the Dirac brackets associated with constraints (3.2).

We recall that the Dirac brackets can be calculated as follows. If $\Phi = \{\phi_\alpha\}_{\alpha\in I}$ denotes the set of all the above constraints, we have

$$\Delta_{\alpha\beta} = \{\phi_\alpha, \phi_\beta\} \text{ is invertible: } \sum_{\gamma\in I} \Delta_{\alpha\gamma}\bar{\Delta}^{\gamma\beta} = \delta^\beta_\alpha \qquad \text{where} \quad \bar{\Delta}^{\alpha\beta} \equiv (\Delta^{-1})_{\alpha\beta}. \tag{3.3}$$

The Dirac brackets are constructed as

$$\{X, Y\}_* \sim \{X, Y\} - \sum_{\alpha,\beta\in I} \{X, \phi_\alpha\}\bar{\Delta}^{\alpha\beta}\{\phi_\beta, Y\} \qquad \forall X, Y \tag{3.4}$$

where the symbol $\sim$ means that one has to apply the constraints on the right-hand side *once the PB have been computed*.

Dirac brackets are designed in such a way that

$$\{A, \varphi_\alpha\}_* = 0 \qquad \forall\alpha \in I \quad \forall A \tag{3.5}$$

Note that, owing to its construction, the Dirac bracket fulfils the conditions required for PB as soon as the original PB does. In other words, it is graded antisymmetric, and obeys the graded Leibniz rule and the graded Jacobi identity, where the $\mathbb{Z}_2$-grade [.] is that introduced in (2.1):

$$\{A, B\}_* = -(-1)^{[A][B]}\{B, A\}_* \tag{3.6}$$

$$\{A, BC\}_* = \{A, B\}_* C + (-1)^{[A][B]} B\{A, C\}_* \tag{3.7}$$

$$\{A, \{B, C\}_*\}_* = \{\{A, B\}_*, C\}_* + (-1)^{[A][B]}\{B, \{A, C\}_*\}_*. \tag{3.8}$$

These identities can be shown by a simple (although a bit lengthy) calculation.

3.2. Soldering procedure

The soldering procedure is an alternative way to compute the PB of $\mathcal{W}(\mathcal{G}, \mathcal{H})$ algebras. We apply it to the superalgebra $gl(Mp|Np)$ with generators $\mathcal{E}^{jm}_{ab}$, $0 \leqslant j \leqslant p-1$, $-j \leqslant m \leqslant j$, $a, b = 1, \ldots, M+N$ (see appendix A). Let M^{jm}_{ab} be the $(M+N)$ square matrices representing the generators $\mathcal{E}^{jm}_{ab}$ in the fundamental representation of $gl(Mp|Np)$. Denoting by J^{ab}_{jm} the dual basis, we introduce the matrix

$$\mathbb{J} \equiv \sum_{a,b=1}^{M+N}\sum_{j=0}^{p-1}\sum_{m=-j}^{j} J^{ab}_{jm} M^{jm}_{ab}. \tag{3.9}$$

Let us consider an infinitesimal transformation of parameters λ^{ab}_{jm}. For convenience, we define the matrix $\lambda \equiv \lambda^{ab}_{jm} M^{jm}_{ab}$,

$$\delta_\lambda \mathbb{J} \equiv \left(\delta_\lambda J^{ab}_{jm}\right) M^{jm}_{ab} = [\lambda, \mathbb{J}] = \{\mathrm{str}(\lambda\mathbb{J}), \mathbb{J}\} \tag{3.10}$$

$$= \lambda^{ef}_{rs} \mathrm{str}\left(M^{rs}_{ef} M^{tu}_{cd}\right) \left\{J^{cd}_{tu}, J^{ab}_{jm}\right\} M^{jm}_{ab} \tag{3.11}$$

where summation over repeated indices is assumed, $[. , .]$ denotes the commutator of $\mathbb{Z}_2$-graded matrices, and $\{. , .\}$ the PB.

We ask $\mathbb{J}$ to be of the form

$$\mathbb{J}|_{g.f.} = \epsilon_- + \sum_{a,b=1}^{M+N} \sum_{j=0}^{p-1} W^{ab}_j M^{jj}_{ab} \tag{3.12}$$

where ϵ_- is the $sl(2)$ negative root generator (see appendix A1). This amounts to constraining the generators J^{ab}_{jm} to obey the following second-class constraints:

$$J^{ab}_{jm} = \delta_{j,1}\delta_{m+1,0}\delta^{ab} \qquad \text{for} \quad -j \leqslant m < j \quad \forall j \quad \forall a, b. \tag{3.13}$$

We look for transformations leaving $\mathbb{J}|_{g.f.}$ with the same form:

$$\delta_\lambda(\mathbb{J}|_{g.f.}) = [\lambda, \mathbb{J}|_{g.f.}] = \left(\delta_\lambda W^{ab}_j\right) M^{jj}_{ab}. \tag{3.14}$$

The parameters λ^{jm}_{ab} are constrained and only $(M+N)^2 p$ of them are free. Equation (3.14) leads to

$$\lambda^{ab}_{j,m+1} = \sum_{k,r=0}^{p-1} \sum_{l=-k}^{k} \sum_{e=1}^{M+N} \left(\lambda^{ae}_{kl} W^{eb}_r \langle k, l; r, r|jm\rangle - W^{ae}_r \lambda^{eb}_{kl} \langle r, r; k, l|jm\rangle\right)$$
$$\text{for} \quad -j \leqslant m \leqslant j-1 \tag{3.15}$$

$$\delta_\lambda W^{ab}_j = \sum_{k,r=0}^{p-1} \sum_{l=-k}^{k} \sum_{e=1}^{M+N} \left(\lambda^{ae}_{kl} W^{eb}_r \langle k, l; r, r|jj\rangle - W^{ae}_r \lambda^{eb}_{kl} \langle r, r; k, l|jj\rangle\right) \tag{3.16}$$

where $\langle\cdot|\cdot\rangle$ are real numbers defined in appendix A1. All the coefficients λ_{kl} can be expressed in terms of the parameters $\lambda_{k,-k}$ and the generators W, after a straightforward but tedious use of equations (3.15).

On the other hand, we have

$$\delta_\lambda W^{ab}_j = \lambda^{ef}_{rs}\ \mathrm{str}\left(M^{rs}_{ef} M^{kk}_{cd}\right) \left\{W^{cd}_k, W^{ab}_j\right\}. \tag{3.17}$$

With appendix A of [8] we obtain

$$\mathrm{str}\left(M^{rs}_{ef} M^{kk}_{cd}\right) = \delta^{rk}\delta^{s,-k}\delta_{fc}\delta_{ed}(-1)^{[d]}(-1)^k(2k)!(k!)^2 \binom{p+k}{2k+1}. \tag{3.18}$$

We define

$$\tilde{\lambda}^{ab}_k \equiv (-1)^k (2k)!(k!)^2 \binom{p+k}{2k+1} \lambda^{ab}_{k,-k}. \tag{3.19}$$

Equation (3.17) becomes

$$\delta_\lambda W^{ab}_j = \sum_{k=0}^{p-1} \sum_{c,d=1}^{M+N} (-1)^{[d]} \tilde{\lambda}^{dc}_k \left\{W^{cd}_k, W^{ab}_j\right\}. \tag{3.20}$$

If we now compare (3.16) and (3.20), the $\tilde{\lambda}_k^{ab}$ being independent of one another, we get $\left\{W_k^{cd}, W_j^{ab}\right\}$ as a polynomial in the Ws.

3.3. Calculation of Poisson brackets

We now give two examples of PB calculations which will be needed in the following.

3.3.1. Calculation of $\left\{W_0^{ab}, W_j^{cd}\right\}$. For $j = 0$ equation (3.16) becomes

$$\begin{aligned}\delta_\lambda W_0^{ab} &= \sum_{k=0}^{p-1} \left(\lambda_{k,-k}^{ae} W_k^{eb} \langle k, -k; k, k|0, 0\rangle - W_k^{ae}\lambda_{k,-k}^{eb}\, \langle k, k; k, -k|0, 0\rangle\right) \\ &= \frac{1}{p}\sum_{k=0}^{p-1}\left(\tilde{\lambda}_k^{ae} W_k^{eb} - (-1)^{([a]+[e])([e]+[b])}\tilde{\lambda}_k^{eb} W_k^{ae}\right). \end{aligned} \tag{3.21}$$

We rewrite equation (3.20) as

$$\begin{aligned}\delta_\lambda W_0^{ab} &= \sum_{k=0}^{p-1}\sum_{c,d=1}^{M+N} (-1)^{[d]}\tilde{\lambda}_k^{dc}\left\{W_k^{cd}, W_0^{ab}\right\} \\ &= \sum_{k=0}^{p-1}\sum_{c,d=1}^{M+N} (-1)^{[d]}(-1)^{1+([a]+[b])([c]+[d])}\tilde{\lambda}_k^{dc}\left\{W_0^{ab}, W_k^{cd}\right\}. \end{aligned} \tag{3.22}$$

Comparing the $\tilde{\lambda}_k^{dc}$-components of both equations, we obtain

$$(-1)^{([a]+[b])([c]+[d])+[d]}\left\{W_0^{ab}, W_k^{cd}\right\} = \frac{1}{p}\left(\delta^{bc}(-1)^{([a]+[d])([d]+[c])} W_k^{ad} - \delta^{ad} W_k^{cb}\right). \tag{3.23}$$

If we define $\hat{W}_k^{ab} \equiv (-1)^{[a]} W_k^{ab}$, $\forall k$, equation (3.23) becomes

$$\left\{\hat{W}_0^{ab}, \hat{W}_k^{cd}\right\} = \frac{1}{p}\left(\delta^{cb}\hat{W}_k^{ad} - \delta^{ad}(-1)^{([a]+[b])([c]+[d])}\,\hat{W}_k^{cb}\right). \tag{3.24}$$

3.3.2. Calculation of $\left\{W_1^{ab}, W_j^{cd}\right\}$. Using the same procedure with $j = 1$ we get

$$\begin{aligned}\delta_\lambda W_1^{ab} &= (-1)^{1+[d]+([a]+[b])([c]+[d])}\left\{W_1^{ab}, W_r^{cd}\right\} \\ \delta_\lambda W_1^{ab} &= \frac{3}{p(p^2-1)}\sum_{k=1}^{p-1}\frac{k(p^2-k^2)}{2k+1}[\tilde{\lambda}_{k-1}, W_k]_-^{ab} + \frac{3}{p(p^2-1)}\sum_{k=1}^{p-1}\sum_{n\geqslant k}^{p-1}[[\tilde{\lambda}_n, W_{n-k}]_-, W_k]_+^{ab} \\ &\quad + \frac{3}{p(p^2-1)}\sum_{k=0}^{p-1}\frac{1}{2k+1}\sum_{n\geqslant k+1}^{p}[[\tilde{\lambda}_{n-1}, W_{n-1-k}]_+, W_k]_-^{ab} \\ &\quad - \frac{3}{p(p^2-1)}\sum_{n\geqslant m>k\geqslant 0}^{p-1}\frac{1}{m(2k+1)}[[[\tilde{\lambda}_n, W_{n-m}]_-, W_{m-1-k}]_-, W_k]_-^{ab}\end{aligned} \tag{3.25}$$

where

$$[\tilde{\lambda}_x, W_y]_\pm^{ab} \equiv \sum_{e=1}^{M+N}\left(\tilde{\lambda}_x^{ae} W_y^{eb} \pm W_y^{ae}\tilde{\lambda}_x^{eb}\right).$$

We use $\hat{W}_k^{ab} \equiv (-1)^{[a]} W_k^{ab}$ and identify the $\tilde{\lambda}_k^{dc}$-components on both sides of the equation,

$$\begin{aligned}
\frac{p(p^2-1)}{3}\big\{\hat{W}_1^{ab}, \hat{W}_r^{cd}\big\} &= \frac{(r+1)(p^2-(r+1)^2)}{2(r+1)+1}\big(\delta^{bc}\hat{W}_{r+1}^{ad} - (-1)^{([a]+[b])([c]+[d])}\delta^{ad}\hat{W}_{r+1}^{cb}\big) \\
&\quad + \sum_{k=1}^{r}\big\{\delta^{bc}(-1)^{[e]}\hat{W}_k^{ae}\hat{W}_{r-k}^{ed} - (-1)^{([a]+[b])([c]+[d])}\delta^{ad}(-1)^{[e]}\hat{W}_{r-k}^{ce}\hat{W}_k^{eb} \\
&\quad + (-1)^{[b]([c]+[d])+[c][d]}\big(\hat{W}_{r-k}^{ad}\hat{W}_k^{cb} - \hat{W}_k^{ad}\hat{W}_{r-k}^{cb}\big)\big\} \\
&\quad + \sum_{k=0}^{r-1}\frac{r-k}{2k+1}\big\{\delta^{bc}(-1)^{[e]}\hat{W}_k^{ae}\hat{W}_{r-k}^{ed} - (-1)^{([a]+[b])([c]+[d])}\delta^{ad}(-1)^{[e]}\hat{W}_{r-k}^{ce}\hat{W}_k^{eb} \\
&\quad + (-1)^{[b]([c]+[d])+[c][d]}\big(\hat{W}_k^{ad}\hat{W}_{r-k}^{cb} - \hat{W}_{r-k}^{ad}\hat{W}_k^{cb}\big)\big\} \\
&\quad - \sum_{r\geqslant m>k\geqslant 0}^{p-1}\frac{1}{m(2k+1)}\big\{\delta^{cb}(-1)^{[e]+[f]}\hat{W}_k^{ae}\hat{W}_{m-k-1}^{ef}\hat{W}_{r-m}^{fd} \\
&\quad - (-1)^{([a]+[b])([c]+[d])}\delta^{ad}(-1)^{[e]+[f]}\hat{W}_{r-m}^{ce}\hat{W}_{m-k-1}^{ef}\hat{W}_k^{fb} + (-1)^{[b]([c]+[d])+[c][d]} \\
&\quad \times\big(\hat{W}_{r-m}^{ad}(-)^{[e]}\big(\hat{W}_{m-k-1}^{ce}\hat{W}_k^{eb}\big) - (-1)^{[e]}\big(\hat{W}_k^{ae}\hat{W}_{m-k-1}^{ed}\big)\hat{W}_{r-m}^{cb} \\
&\quad + \hat{W}_{m-k-1}^{ad}(-)^{[e]}\big(\hat{W}_{r-m}^{ce}\hat{W}_k^{eb}\big) - (-1)^{[e]}\big(\hat{W}_k^{ae}\hat{W}_{r-m}^{ed}\big)\hat{W}_{m-k-1}^{cb} \\
&\quad + \hat{W}_k^{ad}(-)^{[e]}\big(\hat{W}_{r-m}^{ce}\hat{W}_{m-k-1}^{eb}\big) - (-1)^{[e]}\big(\hat{W}_{m-k-1}^{ae}\hat{W}_{r-m}^{ed}\big)\hat{W}_k^{cb}\big)\big\}
\end{aligned} \tag{3.26}$$

where summation over $e, f, g = 1, \ldots, M+N$ is assumed. We recall that

$$\hat{W}_j^{ab} = (-1)^{[a]} W_j^{ab}. \tag{3.27}$$

The $\hat{W}$-basis is the one we will work on, we shall therefore omit the ˆ on W from now on.

3.4. $\mathcal{W}(sl(Mp|Np), (M+N)sl(p))$ superalgebras

The $sl(2)$ principal embedding in $(M+N)gl(p)$ is indeed an embedding in $(M+N)sl(p)$, i.e. it commutes with the $(M+N)gl(1)$ generators defined by $gl(p) = sl(p) \oplus gl(1)$. Moreover, considering these $(M+N)gl(1)$ subalgebras in $gl(Mp|Np)$ which commutes with $(M+N)sl(p)$, it is easy to see that none of its generators is affected by constraints (3.13), since they are highest weights. Furthermore, these $gl(1)$ generators, while they do not commute with all the constraints, weakly commute with them. By weakly, we mean after using the constraints (once the PB have been computed). Thus, their Dirac brackets coincide with their original PB. This implies that these $gl(1)$ generators still form $gl(1)$ subalgebras in the $\mathcal{W}$-superalgebra.

In addition, the diagonal $gl(1)$ of these $(M+N)gl(1)$ subalgebras, which corresponds to the decomposition $gl(Mp|Np) = sl(Mp|Np) \oplus gl(1)$, is central for the original PB. Therefore, this $gl(1)$ generator is still central for the Dirac brackets. In other words, one gets

$$\begin{aligned}
\mathcal{W}_p(M|N) &= \mathcal{W}(gl(Mp|Np), (M+N)gl(p)) \\
&= \mathcal{W}(gl(Mp|Np), (M+N)sl(p)) \\
&= \mathcal{W}[sl(Mp|Np) \oplus gl(1), (M+N)sl(p)] \\
&= \mathcal{U}(\mathcal{W}[sl(Mp|Np), (M+N)sl(p)] \oplus gl(1)).
\end{aligned}$$

4. Truncated super-Yangians and $\mathcal{W}$-superalgebras

4.1. $\mathcal{W}_p(M|N)$ as a deformation of a truncated polynomial algebra

Property 4.1. *The $\mathcal{W}_p(M|N)$ superalgebra is a deformation of the truncated polynomial superalgebra $gl(M|N)_p$.*

Proof. To see that the $\mathcal{W}_p(M|N)$ is a deformation of a truncated polynomial algebra based on $gl(M|N)$, we modify the constraints to

$$\mathbb{J} = \frac{1}{\hbar}\epsilon_- + \sum_{a,b=1}^{N} \sum_{j=0}^{p-1} \sum_{0 \leqslant m \leqslant j} J_{jm}^{ab} M_{ab}^{jm}. \tag{4.1}$$

These constraints are equivalent to the previous ones as soon as $\hbar \neq 0$ (they correspond to a rescaling $J_{jm}^{ab} \to \hbar^{-m} J_{jm}^{ab}$). With these new constraints, the equations associated with the soldering procedure read

$$\lambda_{j,m+1}^{ab} = \hbar \sum_{k,r=0}^{p-1} \sum_{l=-k}^{k} \sum_{e=1}^{M+N} \left(\lambda_{kl}^{ae} W_r^{eb} \langle k,l;r,r|jm\rangle - W_r^{ae} \lambda_{kl}^{eb} \langle r,r;k,l|jm\rangle \right)$$
$$\text{for} \quad -j \leqslant m \leqslant j-1 \tag{4.2}$$
$$\delta_\lambda W_j^{ab} = \sum_{k,r=0}^{p-1} \sum_{l=-k}^{k} \sum_{e=1}^{M+N} \left(\lambda_{kl}^{ae} W_r^{eb} \langle k,l;r,r|jj\rangle - W_r^{ae} \lambda_{kl}^{eb} \langle r,r;k,l|jj\rangle \right).$$

This implies that the parameter $\lambda_{j,m}^{ab}$ behaves as $\hbar^{j+m}$. Then, the Poisson brackets of the W generators take the form

$$\left\{ W_j^{ab}, W_\ell^{cd} \right\}_\hbar = \delta^{bc} W_{j+\ell}^{ad} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad} W_{j+\ell}^{cb} - \hbar P_\hbar^{abcd}(W) \tag{4.3}$$

where $P_\hbar^{abcd}(W)$, polynomial in the Ws, has only positive (or null) powers of $\hbar$. This clearly shows that the $\mathcal{W}_p(M|N)$ superalgebra is a deformation of the superalgebra generated by $W_j^{ab} \equiv J_{jj}^{ab}$ and with defining (undeformed) Poisson brackets:

$$\left\{ W_j^{ab}, W_\ell^{cd} \right\}_0 = \delta^{bc} W_{j+\ell}^{ad} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad} W_{j+\ell}^{cb} \qquad \text{if} \quad j+\ell < p \tag{4.4}$$

$$= 0 \qquad \text{if} \quad j+\ell \geqslant p \tag{4.5}$$

One recognizes in this superalgebra a (enveloping) polynomial algebra based on $gl(M|N)$ quotiented by the relations $W_j^{ab} = 0$ if $j \geqslant p$. In other words, this algebra is nothing but a truncated polynomial algebra, and the $\mathcal{W}$-superalgebra is a deformation of it. □

Property 4.2. *There exist two sets of generators $\left\{ {}^\pm\bar{W}_j^{ab} \right\}_{j=0,\ldots}$ in $\mathcal{W}_p(M|N)$ such that, $\forall a,b,c,d = 1,\ldots,M+N$,*

$$\forall j \geqslant 1 \qquad \left\{ {}^\pm\bar{W}_1^{ab}, {}^\pm\bar{W}_j^{cd} \right\} = \delta^{cb}\, {}^\pm\bar{W}_{j+1}^{ad} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad}\, {}^\pm\bar{W}_{j+1}^{cb}$$
$$+ (-1)^{[c]([a]+[b])+[a][b]} \left(\bar{W}_0^{cb}\, {}^\pm\bar{W}_j^{ad} - {}^\pm\bar{W}_j^{cb}\, \bar{W}_0^{ad} \right) \tag{4.6}$$

$$\forall j \geqslant 0 \qquad \left\{ \bar{W}_0^{ab}, {}^\pm\bar{W}_j^{cd} \right\} = \delta^{cb}\, {}^\pm\bar{W}_j^{ad} - (-1)^{([a]+[b])([c]+[d])} \delta^{ad}\, {}^\pm\bar{W}_j^{cb}. \tag{4.7}$$

The generators ${}^\pm\bar{W}_j^{ab}$ are polynomials of degree $(j+1)$ in the original generators W_j^{ab} and are recursively defined by

$$\bar{W}_0^{ab} \equiv {}^+\bar{W}_0^{ab} = {}^-\bar{W}_0^{ab} = p W_0^{ab} \tag{4.8}$$

$$ {}^{\pm}\bar{W}_1^{ab} = \pm\frac{p(p^2-1)}{6}W_1^{ab} + \frac{p(p\pm 1)}{2}\sum_{e=1}^{M+N}(-1)^{[e]}W_0^{ae}W_0^{eb} \tag{4.9} $$

and for $j > 1$,

$$ {}^{\pm}\bar{W}_j^{ab} = \sum_{n=1}^{j+1}\sum_{|\vec{s}|=j+1-n} {}^{\pm}\alpha_{\vec{s}}^{n,j} \sum_{i_1,\ldots,i_{n-1}=1}^{M+N}(-1)^{[i_1]+\cdots+[i_{n-1}]}\, W_{s_1}^{ai_1}W_{s_2}^{i_1i_2}\cdots W_{s_n}^{i_{n-1}b} \tag{4.10} $$

for some numbers ${}^{\pm}\alpha_{\vec{s}}^{n,j}$ *determined by (4.6). The summation on* $\vec{s}$ *is understood as a summation on n positive (or null) integers* $(s_1,\ldots,s_n)\equiv\vec{s}$ *such that* $|\vec{s}|\equiv\sum_{i=1}^n s_i = j+1-n$.

The subsets $\{{}^{\pm}\bar{W}_j^{ab}\}_{j=0,\ldots,p-1}$ *form two bases of* $\mathcal{W}_p(M|N)$, *the other generators* $\{{}^{\pm}\bar{W}_j^{ab}\}_{j\geqslant p}$ *being polynomials in the basis elements.*

Proof. As in [8] relations (4.6) and (4.7) can be proved by recursion on j. Indeed, a direct calculation shows that (4.7) is obeyed by (4.10) for any numbers ${}^{\pm}\alpha_{\vec{s}}^{n,j}$. Then, (4.6) uniquely determine these numbers, up to the choice made in (4.9). □

Remark 1. Relations (4.6) allow us to compute recursively all the PB of $\mathcal{W}_p(M|N)$ but $\{{}^{\pm}\bar{W}_j^0, {}^{\pm}\bar{W}_k^0\}$, where

$$ {}^{\pm}\bar{W}_j^0 = \sum_{a=1}^{M+N} {}^{\pm}\bar{W}_j^{aa}. \tag{4.11} $$

In the following, we will assume that

$$ \{{}^{\pm}\bar{W}_j^0, {}^{\pm}\bar{W}_k^0\} = 0 \qquad \forall j,k. \tag{4.12} $$

Note that (4.6) and (4.7) prove that (4.12) is valid for $j=0,1$ and $\forall k$. Let us also remark that, since $\mathcal{W}_p(M|N)$ is a deformation of $gl(M|N)$ (see below), the lemma B.1 ensures that $\{{}^{\pm}\bar{W}_j^0, {}^{\pm}\bar{W}_k^0\}$ is central in $\mathcal{W}_p(M|N)$.

The first and the last coefficients that appear in definition (4.10) can be computed by recursion ($\forall j\geqslant 0$):

$$ {}^{\pm}\alpha_j^{1,j} = (\pm 1)^j (j!)^2\binom{p+j}{2j+1} \tag{4.13} $$

$$ {}^{-}\alpha_{(0,\ldots,0)}^{j,j+1} = \binom{p}{j+1} \tag{4.14} $$

$$ {}^{+}\alpha_{(0,\ldots,0)}^{j,j+1} = \binom{p+j}{j+1}. \tag{4.15} $$

The non-vanishing coefficients (4.13) show that the generators ${}^{\pm}\bar{W}_j^{ab}$ for $j<p$ are indeed independent, since these generators write ${}^{\pm}\bar{W}_j^{ab} = {}^{\pm}\alpha_j^{1,j}W_j^{ab}$ + lower, where lower is a polynomial in W_k with $k<j$.

Corollary 4.3. *The change of generators between* $\{{}^{+}\bar{W}_j^{ab}\}_{j=1,\ldots}$ *and* $\{{}^{-}\bar{W}_j^{ab}\}_{j=1,\ldots}$ *is given by*

$$ {}^{\pm}\bar{W}_j^{ab} = \sum_{n=1}^{j+1}(-1)^{j+1+n}\sum_{|\vec{s}|=j+1-n}\;\sum_{i_1,\ldots,i_{n-1}=1}^{M+N} {}^{\mp}\bar{W}_{s_1}^{ai_1}\cdots {}^{\mp}\bar{W}_{s_n}^{i_{n-1}b}(-1)^{[i_1]+\cdots+[i_{n-1}]}. \tag{4.16} $$

Proof. The procedure is the same as in [8]: a direct calculation shows that indeed expression (4.16) satisfies (4.6), (4.7), and that (4.16) is valid for ${}^{\pm}\bar{W}_1^{ab}$. □

Corollary 4.4. *The basis* $\{{}^{-}\bar{W}_j^{ab}\}_{j=1,\ldots,p-1}$ *is such that* ${}^{-}\bar{W}_j^{ab} = 0$ *for* $j \geqslant p$. *In the basis* $\{{}^{+}\bar{W}_j^{ab}\}_{j=1,\ldots,p-1}$ *all the* ${}^{+}\bar{W}_j^{ab}$ *generators* ($j \geqslant p$) *are non-vanishing.*

Proof. (4.15) shows that ${}^{+}\bar{W}_j^{ab} \neq 0$ for $j \geqslant p$. Now, using (4.6) for $j = p$, with the form (4.10), one gets $\alpha_{\frac{n}{s}}^{n,p} = (-1)^n A$ with $A = 0$ or 1. Then, (4.14) shows that $A = 0$ for ${}^{-}\bar{W}_p^{ab}$. Finally, (4.6) ensures that ${}^{-}\bar{W}_j^{ab} = 0$, for $j > p$, as soon as ${}^{-}\bar{W}_p^{ab} = 0$. □

4.2. $\mathcal{W}_p(M|N)$ and $Y_p(M|N)$

We have shown that both $\mathcal{W}_p(M|N)$ and $Y_p(M|N)$ are deformations of a truncated polynomial superalgebra based on $gl(M|N)$. It remains to show that these deformations coincide.

Theorem 4.5. *The* $\mathcal{W}_p(M|N)$ *superalgebra is the truncated super-Yangian* $Y_p(M|N)$.

Proof. First, the map ${}^{-}\bar{W}_j^{ab} \to T_{j-1}^{ab}, \forall 0 \leqslant j < p$, between basis vectors shows that $\mathcal{W}_p(M|N)$ and $Y_p(M|N)$ are isomorphic as vector spaces (and indeed coincide with $gl(M|N)$). Since they are both deformations of $gl(M|N)_p$, we can introduce φ^W and φ^T, the cochains associated with the deformation corresponding to $\mathcal{W}_p(M|N)$ and $Y_p(M|N)$ respectively.

Now, remark that the two superalgebras have identical (in fact undeformed) PB on the couples $({}^{-}\bar{W}_0^{ab}, {}^{-}\bar{W}_j^{cd})$, which proves that the cochains φ^W and φ^T coincide (in fact vanish) on these points. It is also the case for the couples $({}^{-}\bar{W}_j^{0}, {}^{-}\bar{W}_k^{0})$, due to formula (2.27) and assumption (4.12).

Moreover, property 4.2 shows that the cochains φ^W and φ^T coincide on the couples $({}^{-}\bar{W}_1^{ab}, {}^{-}\bar{W}_j^{cd})$. Since φ^W and φ^T are cocycles, this is enough (using lemma B.1) to prove that they are identical. □

4.3. Representations of $\mathcal{W}_p(M|N)$

Theorem 4.6. *Any finite-dimensional irreducible representation of the* $\mathcal{W}_p(M|N)$ *superalgebra is highest weight. It has a unique (up to scalar multiplication) highest weight vector.*

Proof. An irreducible representation π of the $\mathcal{W}_p(M|N)$ superalgebra can be lifted to a representation of the whole super-Yangian by setting $\pi(T_{(r)}^{ij}) = 0$ for $r > n$. It is then obviously irreducible for the super-Yangian, and thus is highest weight by theorem 2.2. □

Theorem 4.7 (Finite dimensional irreducible representations of ($\mathcal{W}_p(M|N)$)**.** *Any finite-dimensional irreducible representation of the* $\mathcal{W}_p(M|N)$ *superalgebra is isomorphic to an evaluation representation or to the subquotient of tensor product of at most p evaluation representations.*

Proof. By evaluation representations for $\mathcal{W}_p(M|N)$ superalgebra, we mean definitions 2.4 and 2.5 with the change $T_r^{ab} \to W_{r-1}^{ab}$ (i.e. the evaluation representations of the truncated super-Yangian). Property (2.21) clearly shows that the (subquotient of) tensor product of n evaluation representations is a representation of the truncated super-Yangian as soon as $n \leqslant p$. It also shows that if it is irreducible for the super-Yangian, then it is also irreducible for the truncated super-Yangian and that they are finite dimensional.

Now conversely, an irreducible representation π of the $\mathcal{W}_p(M|N)$ superalgebra can be lifted to a representation of the whole super-Yangian by setting $\pi\big(T^{ij}_{(r)}\big) = 0$ for $r > n$. It is then obviously irreducible for the super-Yangian, and thus is isomorphic to the (irreducible subquotient of) tensor product of evaluation representations. □

5. Twisted super-Yangians

Twisted super-Yangian have been introduced in [17]. We recall here the main results.

We start with the super-Yangian $Y(M|2n)$, and introduce the transposition t on matrices,

$$E^t_{ab} = (-1)^{[a]([b]+1)}\theta_a\theta_b E_{\bar{b}\bar{a}} \qquad \text{with} \quad \begin{cases} \bar{a} = M+1-a & \text{for} \quad 1 \leqslant a \leqslant M \\ \bar{a} = 2M+2n+1-a & \text{for} \quad M < a \leqslant M+2n \end{cases} \tag{5.1}$$

where the θ_a are given by

$$\begin{aligned} &\theta_a = 1 \qquad \text{for} \quad 1 \leqslant a \leqslant M \\ &\theta_a = \mathrm{sg}\left(\tfrac{2M+2n+1}{2} - a\right) \qquad \text{for} \quad M+1 \leqslant a \leqslant M+2n. \end{aligned} \tag{5.2}$$

Note that we have the relations

$$(-1)^{[a]}\theta_a\theta_{\bar{a}} = 1 \qquad \text{and} \qquad [a] = [\bar{a}] \qquad \forall a. \tag{5.3}$$

Then, we define on $Y(M|2n)$

$$\tau[T(u)] = \sum_{a,b} \tau[T^{ab}(u)]E_{ab} = \sum_{a,b} T^{ab}(-u)E^t_{ab} \tag{5.4}$$

which for the super-Yangian generators reads

$$\tau(T^{ab}(u)) = (-1)^{[a]([b]+1)}\theta_a\theta_b T^{\bar{b}\bar{a}}(-u) \tag{5.5}$$

where τ is an algebra automorphism of $Y(M|2n)$.

One defines in $Y(M|2n)$

$$S(u) = T(u)\tau[T(u)] = \sum_{a,b=1}^{M+N} S^{ab}(u)E_{ab} = \mathbb{I} + \sum_{a,b=1}^{M+N}\sum_{n>0} u^{-n} S^{ab}_{(n)}E_{ab} \tag{5.6}$$

$$S^{ab}_{(n)} = \sum_{c=1}^{M+N}\sum_{p=0}^{n} (-1)^p(-1)^{[c]([b]+1)}\theta_c\theta_b T^{ac}_{(n-p)}T^{\bar{b}\bar{c}}_{(p)} \tag{5.7}$$

$$S^{ab}(u) = \sum_{c=1}^{M+N} (-1)^{[c]([b]+1)}\theta_c\theta_b T^{ac}(u)T^{\bar{b}\bar{c}}(-u). \tag{5.8}$$

Definition 5.1. *$S(u)$ defines a subalgebra of the super-Yangian, the twisted super-Yangian $Y(M|2n)^+$. It obeys the following relation,*

$$R_{12}(u-v)S_1(u)R'_{12}(u+v)S_2(v) = S_2(v)R'_{12}(u+v)S_1(u)R_{12}(u-v) \tag{5.9}$$

where $R(x)$ is the super-Yangian R-matrix,

$$R'(x) = \mathbb{I} + \frac{1}{x}Q = R^{t_1}(-x) \qquad \textit{with} \qquad Q = P^{t_1} \tag{5.10}$$

and t_1 is the transposition (5.1) in the first auxiliary space.

Introducing

$$\tau(S(u)) = \sum_{a,b=1}^{M+N} S^{ab}(-u) E^t_{ab} \tag{5.11}$$

one obtains

$$\tau(S^{ab}(u)) = (-1)^{[a]([b]+1)} \theta_a \theta_b S^{\bar{b}\bar{a}}(-u). \tag{5.12}$$

Then, using expression (5.8) and the commutation relations of the super-Yangian, one can show the symmetry relation

$$\tau(S(u)) = S(u) + \frac{\theta_0}{2u}(S(u) - S(-u)). \tag{5.13}$$

Note that relation (5.9) is equivalent to the following commutator,

$$\begin{aligned}[S_1(u), S_2(v)] &= \frac{1}{u-v}(P_{12}S_1(u)S_2(v) - S_2(v)S_1(u)P_{12}) - \frac{1}{u+v}(S_1(u)Q_{12}S_2(v) \\ &\quad - S_2(v)Q_{12}S_1(u)) + \frac{1}{u^2-v^2}(P_{12}S_1(u)Q_{12}S_2(v) - S_2(v)Q_{12}S_1(u)P_{12})\end{aligned} \tag{5.14}$$

and also to

$$\begin{aligned}[S^{ab}(u), S^{cd}(v)\} &= \frac{(-1)^{([a]+[b])[c]}}{u-v}(-1)^{[a][b]}(S^{cb}(u)S^{ad}(v) - S^{cb}(v)S^{ad}(u)) \\ &\quad - \frac{(-1)^{([a]+[b])[c]}}{u+v}\big((-1)^{[a][c]}\theta_b\theta_{\bar{c}}S^{a\bar{c}}(u)S^{\bar{b}d}(v) - (-1)^{[b][d]}\theta_{\bar{a}}\theta_d S^{c\bar{a}}(v)S^{\bar{d}b}(u)\big) \\ &\quad + \frac{(-1)^{([a]+[b])[c]}}{u^2-v^2}(-1)^{[a]}\theta_a\theta_b(S^{c\bar{a}}(u)S^{\bar{b}d}(v) - S^{c\bar{a}}(v)S^{\bar{b}d}(u)).\end{aligned} \tag{5.15}$$

As for $Y(M|N)$, one can show that $Y(M|2n)^+$ is a deformation of $\mathcal{U}(osp(M|2n)[x])$.

5.1. *Finite-dimensional irreducible representations of twisted super-Yangians*

The finite-dimensional irreducible representations of twisted super-Yangians have been studied in [17]. We recall here the main results. As for super-Yangians, they rely on the evaluation morphism:

Property 5.2. *The following map defines an algebra inclusion*

$$\begin{aligned} &Y(M|2n)^+ \to \mathcal{U}[osp(M|2n)] \\ &S(u) \to \mathbb{F}(u) = \mathbb{I} + \frac{1}{u+\frac{1}{2}}F \end{aligned} \tag{5.16}$$

where the $osp(M|2n)$ generators J^{ab} have been gathered in the matrix

$$F = \sum_{a,b=1}^{M+N} J^{ab} F_{ab} \qquad \textit{with} \quad F_{ab} = E_{ab} - (-1)^{[a]([b]+1)}\theta_a\theta_b\, E_{\bar{b}\bar{a}}. \tag{5.17}$$

Using the above inclusion, one constructs from any finite-dimensional irreducible representation of $osp(M|2n)$, a finite-dimensional irreducible representation of $Y(M|2n)^+$.

Theorem 5.3. *Every finite-dimensional irreducible $Y(M|2n)^+$-module contains a unique (up to scalar multiples) highest weight vector.*

A sufficient condition for the existence of irreducible finite-dimensional representations has been given in [17]. It corresponds to an explicit construction of the representation as a tensor product of $Y(M|N)$ evaluation representations and possibly one $osp(M|2n)$ representation (using the evaluation morphism). These sufficient conditions were conjectured to be necessary; we will assume this conjecture in the following.

5.2. Classical twisted super-Yangians

As for super-Yangians, one can introduce a classical (Poisson bracket) version of twisted super-Yangians. The calculation is the same as in section 2.4: one writes $R(u-v) = \mathbb{I} + \hbar r(u-v)$, $R'(u+v) = \mathbb{I} + \hbar r'(u+v)$, and considers the terms in $\hbar$. One obtains

$$\begin{aligned}\{S_1(u), S_2(v)\} &= r_{12}(u-v)S_1(u)S_2(v) - S_2(v)S_1(u)r_{12}(u-v)\\ &\quad + S_2(v)r'_{12}(u+v)S_1(u) - S_1(u)r'_{12}(u+v)S_2(v).\end{aligned} \tag{5.18}$$

In components, this reads

$$\begin{aligned}\{S_{(q)1}, S_{(r)2}\} &= \sum_{s=0}^{\mu-1}[P_{12}S_{(s)1}S_{(r+q-s-1)2} - S_{(r+q-s-1)2}S_{(s)1}P_{12}\\ &\quad + (-1)^{q+s}(S_{(s)1}Q_{12}S_{(r+q-s-1)2} - S_{(r+q-s-1)2}Q_{12}S_{(s)1})]\end{aligned}$$

with $\mu = \min(q, p)$.

Let us remark that the symmetry relation (5.13), in its classical form, takes the form

$$\tau(S(u)) = S(-u) \tag{5.19}$$

because the $T^{ab}(u)$ generators are now $\mathbb{Z}_2$-commuting.

6. Folded $\mathcal{W}$-superalgebras revisited

It is well known that the $gl(M|N)$ superalgebra can be folded (using an outer automorphism) into an orthosymplectic one (see e.g. [20]). In the same way, folded $\mathcal{W}$-superalgebras have been defined[3] in [18], and shown to be $\mathcal{W}$-superalgebras based on orthosymplectic superalgebras.

We present here a different proof of this property, adapted to our purpose, and generalized to the case of the automorphisms presented in section 5. For such a purpose, we use the Dirac bracket definition introduced in section 3.1.

6.1. Automorphism of $gl(Mp|2np)$ and $\mathcal{W}_p(M|N)$

As for the super-Yangian, one introduces an automorphism of $gl(Mp|2np)$ defined by

$$\tau\left(J^{ab}_{jm}\right) = (-1)^{j+1}(-1)^{[a]([b]+1)}\theta^a\theta^b J^{\bar{b},\bar{a}}_{jm} \tag{6.1}$$

where θ^a is defined in (5.2), and $\bar{a}$ is given in (5.1).

To prove that τ is an automorphism of $gl(Mp|2np)$, we need the following property of the Clebsch–Gordan coefficient, which was proved in [8]. Note that we need this property only for the algebra $gl(p)$, because of the decomposition $gl(Mp|2np) \sim gl(M|2n) \otimes gl(p)$ used here (see appendix A).

Property 6.1. *The Clebsch–Gordan-like coefficients obey the rule*

$$\langle j, m; t, q|r, s\rangle = (-1)^{j+t+r}\langle t, q; j, m|r, s\rangle. \tag{6.2}$$

[3] Strictly speaking, it is the folding of 'affine' $\mathcal{W}$-superalgebras that has been defined in [18], but the folding of finite $\mathcal{W}$-superalgebras can be defined by the same procedure.

Note that in the above formula, it is not the $\mathbb{Z}_2$-grades $[j]$, $[t]$ or $[r]$ that are used, but really j, t and r themselves.

With this property, it is a simple matter of calculation to show that τ defined in (6.1) is an automorphism of $gl(Mp|2np)$.

6.2. *Folding $gl(Mp|2np)$ and $\mathcal{W}_p(M|2n)$*

6.2.1. $gl(Mp|2np)$. One considers the subalgebra $\mathrm{Ker}(\mathbb{I}-\tau)$ in $gl(Mp|2np)$. It is generated by the combinations

$$K^{ab}_{jm} = J^{ab}_{jm} + \tau\left(J^{ab}_{jm}\right) = J^{ab}_{jm} - (-1)^j(-1)^{[a]([b]+1)}\theta^a\theta^b J^{\bar{b}\bar{a}}_{jm} \tag{6.3}$$

which obey the symmetry relation

$$\tau\left(K^{ab}_{jm}\right) = K^{ab}_{jm} \qquad \text{i.e.} \quad K^{ab}_{jm} = (-1)^{j+1}(-1)^{[a]([b]+1)}\theta^a\theta^b K^{\bar{b}\bar{a}}_{jm}. \tag{6.4}$$

Using the PB

$$\left\{J^{ab}_{jm}, J^{cd}_{k\ell}\right\} = \sum_{r=|j-k|}^{j+k}\sum_{s=-r}^{r}\langle j,m;k,\ell|r,s\rangle\Big(\delta^{bc}J^{ad}_{rs} - (-1)^{([a]+[b])([c]+[d])}(-1)^{j+k+r}\delta^{ad}J^{cb}_{rs}\Big)$$

one can compute the commutation relations

$$\begin{aligned}\left\{K^{ab}_{jm}, K^{cd}_{k\ell}\right\} = \sum_{r=|j-k|}^{j+k}\sum_{s=-r}^{r}\langle j,m;k,\ell|r,s\rangle\Big(&\delta^{bc}K^{ad}_{rs} - (-1)^j\theta^a\theta^b(-1)^{[a]([b]+1)}\delta^{\bar{a}c}K^{\bar{b}d}_{rs}\\ &- (-1)^{j+k+r}(-1)^{([a]+[b])([c]+[d])}\Big[\delta^{ad}K^{cb}_{rs} - (-1)^j\theta^a\theta^b(-1)^{[a]([b]+1)}\delta^{\bar{b}d}K^{c\bar{a}}_{rs}\Big]\Big).\end{aligned}$$

After a rescaling of K^{ab}_{jm}, one recognizes the superalgebra $osp(Mp|2np)$.

Looking at the decomposition of the fundamental of $gl(Mp|2np)$ with respect to the principal embedding of $sl(2)$ in $(M+2n).sl(p)$ (see [18, 19] for the technique used here) one shows that the subalgebra $(M+2n).sl(p)$, generated by the J^{aa}_{jm}s, is folded into an $(m+n).sl(p)$ (respectively $(m+n).sl(p)\oplus so(p)$) when $M=2m$ (respectively $M=2m+1$).

In the following, we will denote this subalgebra by $[M.sl(p)]^\tau \oplus n.sl(p)$.

6.2.2. $\mathcal{W}_p(M|2n)$. We are now dealing with the enveloping algebra of $gl(Mp|2np)$ that we denote by $\mathcal{U}[gl(Mp|2np)] \equiv \mathcal{U}(Mp|2np)$. One introduces the coset

$$\begin{aligned}&\mathcal{U}(Mp|2np)^+ \equiv \mathcal{U}(Mp|2np)/\mathcal{K} \qquad \text{where} \quad \mathcal{K} = \mathcal{U}(Mp|2np)\cdot\mathcal{L} \quad \text{with } \mathcal{L} \text{ spanned by}\\ &\qquad J^{ab}_{jm} - \tau\left(J^{ab}_{jm}\right) \quad \forall\, a,b,j,m\\ &\mathcal{W}_p(M|2n)^+ \equiv \mathcal{W}_p(M|2n)/\mathcal{J} \qquad \text{where} \quad \mathcal{J} = \mathcal{W}_p(M|2n)\cdot\mathcal{I} \quad \text{with } \mathcal{I} \text{ spanned by}\\ &\qquad W^{ab}_j - \tau\left(W^{ab}_j\right) \quad \forall\, a,b,j.\end{aligned}$$

We have the property

Property 6.2. *τ is an automorphism of $\mathcal{U}(Mp|2np)$ provided with the Dirac brackets:*

$$\tau\left(\left\{J^{ab}_{jm}, J^{cd}_{kl}\right\}_*\right) = \left\{\tau\left(J^{ab}_{jm}\right), \tau\left(J^{cd}_{kl}\right)\right\}_*. \tag{6.5}$$

Hence, τ is also an automorphism of $\mathcal{W}_p(M|2n)$.

Proof. It is obvious that τ is an automorphism of Poisson brackets on $\mathcal{U}(Mp|2np)$. Moreover, due to the form of the constraints (3.13), τ acts as a relabelling (up to a sign) of the constraints,

$$\tau(\varphi_\alpha) = \epsilon_{\alpha'}\varphi_{\alpha'} \qquad \text{where} \quad \alpha' \equiv \tau(\alpha) \quad \text{and} \quad \epsilon_{\alpha'} = \epsilon_\alpha = \pm 1 \tag{6.6}$$

which shows that $\tau(\Phi) = \Phi$. We also have

$$\tau(\Delta_{\alpha\beta}) = \epsilon_{\alpha'}\epsilon_{\beta'}\Delta_{\alpha'\beta'}. \tag{6.7}$$

This implies that

$$\tau(\{A, \varphi_\alpha\}\Delta^{\alpha\beta}\{\varphi_\beta, B\}) = \{\tau(A), \varphi_{\alpha'}\}\Delta^{\alpha'\beta'}\{\varphi_{\beta'}, \tau(B)\} = \{\tau(A), \varphi_\alpha\}\Delta^{\alpha\beta}\{\varphi_\beta, \tau(B)\}. \tag{6.8}$$

This shows that this automorphism is compatible with the set of constraints Φ and thus τ is an automorphism of the Dirac brackets. □

Corollary 6.3. *The Dirac brackets provide $\mathcal{W}_p(M|2n)^+$ with an algebraic structure.*

Proof. We define on $\mathcal{W}_p(M|2n)^+$ a bracket which is just the previous Dirac bracket restricted to this coset. Since $\mathcal{W}_p(M|2n)^+$ is generated by elements of the form $W + \tau(W)$, we have

$$\begin{aligned}\{W + \tau(W), W' + \tau(W')\}_* &= \{W, W'\} + \{\tau(W), \tau(W')\}_* + \{\tau(W), W'\}_* + \{W, \tau(W')\}_* \\ &= \{W, W'\} + \{\tau(W), W'\}_* + \tau(\{W, W'\}_* + \{\tau(W), W'\}_*).\end{aligned}$$

□

Indeed we have

Property 6.4. *The $\mathcal{W}_p(M|2n)^+$ superalgebra is the $\mathcal{W}[osp(Mp|2np), [M.sl(p)]^\tau \oplus n.sl(p)]$ superalgebra.*

Above, the $[M.sl(p)]^\tau$ (respectively $n.sl(p)$) subalgebra is understood as the subalgebra of the orthogonal (respectively symplectic) algebra in $osp(Mp|2np)$.

Proof. On the coset, we have $J^{ab}_{jm} \equiv \tau\left(J^{ab}_{jm}\right) \equiv 2K^{ab}_{jm}$. We introduce on $\mathcal{U}(Mp|2np)$

$$2D\varphi_\alpha = \varphi_\alpha - \tau(\varphi_\alpha) \qquad 2S\varphi_\alpha = \varphi_\alpha + \tau(\varphi_\alpha). \tag{6.9}$$

Since these generators satisfy $D\varphi_\alpha = -\tau(D\varphi_\alpha)$ and $S\varphi_\alpha = \tau(S\varphi_\alpha)$ and are in $gl(Mp|2np)$, we have

$$\{S\varphi_\alpha, D\varphi_\beta\} \in \mathcal{I} \qquad \text{i.e.} \quad \{S\varphi_\alpha, D\varphi_\beta\} = 0 \quad \text{on} \quad \mathcal{W}_p(M|2n)^+. \tag{6.10}$$

Similarly, we define

$$D\Delta_{\alpha\beta} = \{D\varphi_\alpha, D\varphi_\beta\} \qquad S\Delta_{\alpha\beta} = \{S\varphi_\alpha, S\varphi_\beta\} \tag{6.11}$$

which obey the properties

$$D\Delta_{\alpha\beta} = \epsilon_{\alpha'}\epsilon_{\beta'}D\Delta_{\alpha'\beta'} = -\epsilon_{\alpha'}D\Delta_{\alpha'\beta} = -\epsilon_{\beta'}D\Delta_{\alpha\beta'} \tag{6.12}$$

$$S\Delta_{\alpha\beta} = \epsilon_{\alpha'}\epsilon_{\beta'}S\Delta_{\alpha'\beta'} = \epsilon_{\alpha'}S\Delta_{\alpha'\beta} = \epsilon_{\beta'}S\Delta_{\alpha\beta'} \tag{6.13}$$

$$\Delta_{\alpha\beta} = S\Delta_{\alpha\beta} + D\Delta_{\alpha\beta} \quad \text{on} \quad \mathcal{W}_p(M|2n)^+. \tag{6.14}$$

We will say that a matrix is τ-antisymmetric when it satisfies a relation such as (6.12), and τ-symmetric when it obeys (6.13). τ-antisymmetric matrices are orthogonal to τ-symmetric ones,

$$D\Delta \cdot S\Delta = 0 \qquad \text{since}$$

$$(D\Delta \cdot S\Delta)_{\alpha\beta} = \sum_\gamma D\Delta_{\alpha\gamma}S\Delta_{\gamma\beta} = \sum_{\gamma'} D\Delta_{\alpha\gamma'}S\Delta_{\gamma'\beta} = -\sum_\gamma D\Delta_{\alpha\gamma}S\Delta_{\gamma\beta}$$

where $S\Delta_{\alpha\beta}$ is the matrix of constraints of $osp(Mp|2np)$ reduced with respect to $[M.sl(p)]^\tau \oplus n.sl(p)$. Thus, it is invertible and the associated Dirac brackets define the superalgebra $\mathcal{W}(osp(Mp|2np), [M.sl(p)]^\tau \oplus n.sl(p))$. It remains to show that, on $\mathcal{W}_p(M|2n)^+$, the previously defined Dirac brackets coincide with the latter Dirac brackets.

For that purpose, we use the form $\Delta = \Delta_0(\mathbb{I} + \widehat{\Delta})$, given in [8], where Δ_0 is an invertible τ-symmetric matrix and $\widehat{\Delta}$ is nilpotent (of finite order r). Introducing the τ-symmetrized and antisymmetrized part of $\widehat{\Delta}$, one deduces

$$\Delta^{-1} = \Delta_0^{-1} \sum_{n=0}^{r} (-1)^n (S\widehat{\Delta} + D\widehat{\Delta})^n = \Delta_0^{-1} \sum_{n=0}^{r} (-1)^n ((S\widehat{\Delta})^n + (D\widehat{\Delta})^n) = S\Delta^{-1} + D\Delta^{-1}$$

which shows that $D\Delta$ is also invertible.

On $\mathcal{W}_p(M|2n)^+$, we have

$$\left\{K^{ab}_{(m)}, K^{cd}_{(n)}\right\}_* = \left\{K^{ab}_{(m)}, K^{cd}_{(n)}\right\} - \left\{K^{ab}_{(m)}, D\varphi_\alpha + S\varphi_\alpha\right\} \Delta^{\alpha\beta} \left\{D\varphi_\beta + S\varphi_\beta, K^{cd}_{(n)}\right\} \tag{6.15}$$

$$= \left\{K^{ab}_{(m)}, K^{cd}_{(n)}\right\} - \left\{K^{ab}_{(m)}, S\varphi_\alpha\right\} \Delta^{\alpha\beta} \left\{S\varphi_\beta, K^{cd}_{(n)}\right\} \tag{6.16}$$

$$= \left\{K^{ab}_{(m)}, K^{cd}_{(n)}\right\} - \left\{K^{ab}_{(m)}, S\varphi_\alpha\right\} (S\Delta^{\alpha\beta} + D\Delta^{\alpha\beta}) \left\{S\varphi_\beta, K^{cd}_{(n)}\right\}. \tag{6.17}$$

From the τ-antisymmetry of $D\Delta^{-1}$, we obtain

$$\{., S\varphi_\alpha\} D\Delta^{\alpha\beta} \{S\varphi_\beta, .\} = \{., S\varphi_{\alpha'}\} D\Delta^{\alpha'\beta} \{S\varphi_\beta, .\} = -\{., S\varphi_\alpha\} D\Delta^{\alpha\beta} \{S\varphi_\beta, .\} = 0 \tag{6.18}$$

which leads to the Dirac brackets,

$$\left\{K^{ab}_{(m)}, K^{cd}_{(n)}\right\}_* = \left\{K^{ab}_{(m)}, K^{cd}_{(n)}\right\} - \left\{K^{ab}_{(m)}, S\varphi_\alpha\right\} S\Delta^{\alpha\beta} \left\{S\varphi_\beta, K^{cd}_{(n)}\right\}. \tag{6.19}$$

These Dirac brackets are just those of the $\mathcal{W}(osp(Mp|2np), [M.sl(p)]^\tau \oplus n.sl(p))$ superalgebra, by definition of $S\Delta$. □

7. Folded $\mathcal{W}$-algebras as truncated twisted Yangians

7.1. Classical case

We start with the $\mathcal{W}_p(M|2n)$ superalgebra in the Yangian basis. The Poisson brackets are

$$\{T_{(m)1}, T_{(n)2}\} = \sum_{r=0}^{\min(m,n)-1} (P_{12} T_{(r)1} T_{(m+n-r)2} - T_{(r)2} T_{(m+n-r)1} P_{12}) \tag{7.1}$$

with the convention $T_{(m)} = 0$ for $m > p$. The action of the automorphism τ, both for twisted super-Yangians and folded $\mathcal{W}_p(M|2n)$ superalgebras, reads

$$\tau(T_{(m)}) = (-1)^m T^t_{(m)}. \tag{7.2}$$

However, from the twisted super-Yangian point of view, one selects the generators

$$S_{(m)} = \sum_{r+s=m} (-1)^s T_{(r)} T^t_{(s)}$$

while in the folded $\mathcal{W}$-superalgebra case, one constrains the generators to $T_{(m)} = (-1)^m T^t_{(m)}$. Although the procedures are different (and indeed lead to different generators), we have

Theorem 7.1. *As an algebra, the $\mathcal{W}$-superalgebra $\mathcal{W}(osp(Mp|2np), [M.sl(p)]^\tau \oplus n.sl(p))$ is isomorphic to the truncation (at level p) of the (classical) twisted super-Yangian $Y(M|2n)^+$.*

More precisely, we have the correspondences

$$\begin{aligned} Y_p(2m+1|2n)^+ &\longleftrightarrow \mathcal{W}[osp(2mp+p|2np), (m+n).sl(p) \oplus so(p)] \\ Y_p(2m|2n)^+ &\longleftrightarrow \mathcal{W}[osp(2mp|2np), (m+n).sl(p)]. \end{aligned} \tag{7.3}$$

Proof. We prove this theorem by showing that the Dirac brackets of the folded $\mathcal{W}$-superalgebra coincide with the Poisson brackets (5.18) with the truncation $S_{(m)} = 0$ for $m > p$.

We start with the $\mathcal{W}_p(M|2n)$ superalgebra in the truncated super-Yangian basis,

$$\{T_{(q)1}, T_{(r)2}\} = \sum_{s=0}^{\mu-1}(P_{12}T_{(s)1}T_{(r+q-s-1)2} - T_{(r+q-s-1)2}T_{(s)1}P_{12}) \qquad \text{with} \quad T_{(s)} = 0$$
$$\text{for} \quad s > p \quad \text{and} \quad \mu = \min(q, r, p). \tag{7.4}$$

In this basis, we define

$$2\varphi_{(s)} = T_{(s)} - (-1)^s T^t_{(s)} \quad \text{and} \quad 2K_{(s)} = T_{(s)} + (-1)^s T^t_{(s)}. \tag{7.5}$$

The folding (of the $\mathcal{W}$-superalgebra) corresponds to

$$\varphi_{(s)} = 0 \qquad \text{i.e.} \quad K_{(s)} = (-1)^s K^t_{(s)}. \tag{7.6}$$

It is a simple matter of calculation to get

$$\begin{aligned}2\{K_{(q)1}, K_{(r)2}\} = \sum_{s=0}^{\mu-1}[&P_{12}K_{(s)1}K_{(r+q-s-1)2} - K_{(r+q-s-1)2}K_{(s)1}P_{12}\\ &+(-1)^{q+s}(K_{(s)1}Q_{12}K_{(r+q-s-1)2} - K_{(r+q-s-1)2}Q_{12}K_{(s)1})]\end{aligned} \tag{7.7}$$

that is to say

$$\begin{aligned}2\{K_1(u), K_2(v)\} &= [r_{12}(u-v), K_1(u)K_2(v)]\\ &\quad + K_2(v)r'_{12}(u+v)K_1(u) - K_1(u)r'_{12}(u+v)K_2(v).\end{aligned}$$

These PB are equivalent to relation (5.18) for $S(u) \equiv K\left(\frac{u}{2}\right)$. Constraint (7.6) is then rewritten as $\tau(S(-u)) = S(u)$. Thus, the folded $\mathcal{W}$-superalgebra and the truncated twisted super-Yangian are defined by the same relations. □

7.2. *Quantization and representations of $\mathcal{W}$-superalgebras*

Now that folded $\mathcal{W}$-superalgebras have proved to be truncations of twisted super-Yangians at classical level, there quantization is very simple. It can be identified with the truncated twisted super-Yangian at quantum level,

$$R_{12}(u-v)S_1(u)R'_{12}(u+v)S_2(v) = S_2(v)R'_{12}(u+v)S_1(u)R_{12}(u-v) \tag{7.8}$$

$$\text{with} \quad \begin{cases} R_{12}(x) = \mathbb{I} - \frac{1}{x}P_{12} \qquad R'_{12}(x) = \mathbb{I} - \frac{1}{x}Q_{12} \\ S(u) = \displaystyle\sum_{m=0}^{p} u^{-m}S_{(m)} \qquad S_{(0)} = \mathbb{I}. \end{cases} \tag{7.9}$$

Using the representation classification of twisted super-Yangians given in [17], one can then deduce the classification of irreducible finite-dimensional representations for truncated twisted super-Yangians in the same way as done in [10] for ordinary twisted Yangians. For conciseness, we will only sketch the results. In particular, one obtains the following theorems.

Theorem 7.2. *Any finite-dimensional irreducible representation of the $\mathcal{W}_p(M|2n)^+$ superalgebra is highest weight.*

Proof. Same proof as for theorem 4.6. □

Theorem 7.3. *Any finite-dimensional irreducible representation of the $\mathcal{W}_p(M|2n)^+$ superalgebra is isomorphic to an evaluation representation, or to the (irreducible subquotient of) tensor product of at most $[p/2]$ evaluation representations of $Y(M|2n)$, and possibly an $osp(M|2n)$ representation.*

Proof. Same proof as for twisted Yangians, see [10], using the results given in [17] for $Y(M|2n)^+$. Indeed, as for the $gl(M|N)$ case, one needs to have $S_{(r)} = 0$ for $r \geqslant p$ to get a representation of the $\mathcal{W}$-superalgebra. This constrains the number of evaluation representations allowed to be tensorized (to get a representation). The difference with the $Y(M|N)$ case lies in the quadratic form $S(u) = T(u)\tau(T(-u))$, which lowers the maximum number of terms in the tensor product. The occurrence of an $osp(M|2n)$ representation is due to the classification given in [17]. □

Reasoning as in [10], one can also obtain a condition on the weights of the representation. We omit it here, due to the lack of place.

Remark 2. As for $\mathcal{W}$-algebras based on $so(M)$ and $sp(2n)$, see [10] for more details, one could think that $\mathcal{W}$-superalgebras based on $osp(M|2n)$ are related to super-Yangians based on $osp(M|2n)$ instead of twisted super-Yangians. However, a simple counting (using the method given in [19]) of the generators shows that it is the twisted super-Yangians that have to be considered.

Appendix A. General settings on $gl(Mp|Np)$

A.1. Clebsch–Gordan-like coefficients

We start with the $gl(Mp|Np)$ superalgebra in its fundamental representation, and consider the $sl(2)$ principal embedding in $(M+N)gl(p) \equiv \underbrace{gl(p) \oplus \cdots \oplus gl(p)}_{M+N}$.

In the fundamental representation, one can view $gl(Mp|Np)$ as $gl(p) \otimes gl(M|N)$, so that the generators of the $sl(2)$ can be written as $\epsilon_{\pm,0} \equiv e_{\pm,0} \otimes \mathbb{1}_{M+N}$. The $e_{\pm,0}$ are the generators of the $sl(2)$ algebra principal in $gl(p)$ and verify $[e_0, e_\pm] = \pm e_\pm$ and $[e_+, e_-] = e_0$. The generator $\mathbb{1}_{M+N}$ is the identity generator in $gl(M|N)$.

Under the adjoint action of the $sl(2)$, $gl(p) \otimes gl(M|N)$ can be decomposed into $sl(2)$ multiplets: $M^{ab}_{jm} \equiv M_{jm} \otimes E^{ab}$ with $a, b = 1, \ldots, M+N$; $-j \leqslant m \leqslant j$; $0 \leqslant j \leqslant p-1$.

The M^{jm} are $p \times p$ matrices resulting from the decomposition of $gl(p)$ into $sl(2)$ multiplets. Properties of the M_{jm} are gathered in appendix A of [8].

The E_{ab} are $(M+N) \times (M+N)$ matrices with 1 at position (a, b). They are the graded part of M^{ab}_{jm} which is even if $a + b \equiv 0 \pmod 2$ and odd otherwise.

Following appendix A of [8] we have

$$\left[\epsilon_+, M^{ab}_{jm}\right] = \frac{j(j+1) - m(m+1)}{2} M^{ab}_{j,m+1} \tag{A.1}$$

$$\left[\epsilon_-, M^{ab}_{jm}\right] = M^{ab}_{j,m-1} \tag{A.2}$$

$$\left[\epsilon_0, M^{ab}_{jm}\right] = m M^{ab}_{jm}. \tag{A.3}$$

The product law (in the fundamental representation) reads

$$M^{ab}_{jm} \cdot M^{cd}_{ln} = \delta^{bc} \sum_{r=|j-l|}^{j+l} \sum_{s=-r}^{r} \langle j, m; l, n | r, s \rangle M^{ad}_{rs} \tag{A.4}$$

which leads to the following commutation relations (valid in the abstract algebra):

$$\begin{aligned} \left[M^{ab}_{jm}, M^{cd}_{ln}\right] = \sum_{r=|j-l|}^{j+l} \sum_{s=-r}^{r} \Big(&\delta^{bc} \langle j, m; l, n | r, s \rangle M^{ad}_{rs} \\ &- (-1)^{([a]+[b])([c]+[d])} \delta^{ad} \langle l, n; j, m | r, s \rangle M^{cb}_{rs}\Big). \end{aligned}$$

The scalar product is:

$$\eta^{ab,cd}_{j,m;l,n} = \mathrm{str}\left(M^{ab}_{jm} \cdot M^{cd}_{ln}\right) = (-1)^{[a]}\delta^{ad}\delta^{cb}(-1)^m \delta_{j,\ell}\delta_{m+n,0}\eta_j \tag{A.5}$$

for some non-vanishing coefficient η_j, given in [8].

The 'Clebsch–Gordan-like' coefficients are then given by

$$\langle jm, k\ell | r, s\rangle = \frac{(-1)^s}{\eta_r}\,\mathrm{Tr}(M_{jm}M_{k\ell}M_{r,-s}). \tag{A.6}$$

We recall that in (A.6) it is the usual trace operator which is involved, since we are in the $gl(p)$ Lie algebra.

A.2. Structure constants

We consider a Lie superalgebra $\mathcal{G}$ in its fundamental representation, with homogeneous generators t_a. As usual, we can define a gradation index [ ] such that

$$[a] = \begin{cases} 0 & \text{if } t_a \text{ bosonic} \\ 1 & \text{if } t_a \text{ fermionic.} \end{cases}$$

The commutation relations are $[t_a, t_b\} = f_{ab}{}^c t_c$ (summation over repeated indices). The structure constants have the following property: $f_{ab}{}^c \neq 0 \Rightarrow [a]+[b]+[c] = 0$. They obey the graded Jacobi identity

$$f_{ab}{}^d f_{dc}{}^e = f_{bc}{}^d f_{ad}{}^e + (-1)^{[b][c]} f_{ac}{}^d f_{db}{}^e. \tag{A.7}$$

Note that the adjoint representation for superalgebras takes the form

$$\mathrm{ad}(t_a)^c_b = -(-1)^{[a][b]} f_{ab}{}^c. \tag{A.8}$$

The invariant metric g_{ab} is proportional to $\mathrm{str}_F(t_a t_b)$, where the supertrace is taken in the fundamental representation. Note that the Killing form, which is the supertrace in the adjoint representation, can be degenerate (in fact null) for some superalgebras, e.g. $gl(M|M)$ [20]. The invariant metric has the following properties:

$$g_{ab} = (-1)^{[a][b]} g_{ba} \quad \text{and} \quad g_{ab} = 0 \qquad \text{if} \quad [a] \neq [b]. \tag{A.9}$$

We introduce its inverse g^{ab} and use it to raise and lower the indices. For instance, $t^a \equiv g^{ab} t_b$ and $f^{ab}{}_c \equiv g^{a\alpha} g^{b\beta} f_{\alpha\beta}{}^\gamma g_{\gamma c}$; we therefore have $[t^a, t^b\} = f^{ab}{}_c t^c$.

Defining the tensor $f_{abc} = f_{ab}{}^\gamma g_{\gamma c}$, one can take it totally (graded) antisymmetric,

$$f_{abc} = -(-1)^{[a][b]} f_{bac} = -(-1)^{[b][c]} f_{acb}. \tag{A.10}$$

Appendix B. Deformations and cohomology

Let us consider a Lie superalgebra $\mathcal{A}$ with homogeneous generators u_α and Lie bracket,

$$\{u_\alpha, u_\beta\} = f_{\alpha\beta}{}^\gamma u_\gamma. \tag{B.1}$$

The gradation index [] is such that $[\alpha] = 0$ if u_α is bosonic and $[\alpha] = 1$ if u_α is fermionic.

We aim at constructing a deformation of the Lie bracket (B.1), following e.g. [16].

For such a purpose, we introduce n-cochains ($n \in \mathbb{Z}_{>0}$), i.e. linear maps $\chi^{(n)}$ from $\mathcal{A}^n$ to $\mathcal{A}$ with the following property:

$$\chi^{(n)}\left(u_{\alpha_1}, \ldots, u_{\alpha_i}, u_{\alpha_{i+1}}, \ldots, u_{\alpha_n}\right) = (-1)^{1+[\alpha_i][\alpha_{i+1}]}\chi^{(n)}\left(u_{\alpha_1}, \ldots, u_{\alpha_{i+1}}, u_{\alpha_i}, \ldots, u_{\alpha_n}\right). \tag{B.2}$$

The Chevalley derivation δ maps n-cochains to $(n+1)$-cochains,

$$(\delta\chi^{(n)})\left(u_{\alpha_0}, u_{\alpha_1}, \ldots, u_{\alpha_n}\right) = \sum_{i=0}^{n}(-1)^{i+\epsilon_i}\left\{u_{\alpha_i}, \chi^{(n)}\left(u_{\alpha_0}, \ldots, \hat{u}_{\alpha_i}, \ldots, u_{\alpha_n}\right)\right\}$$
$$+ \sum_{0\leqslant i<j\leqslant n}(-1)^{i+j+\epsilon_{ij}}\chi^{(n)}\left(\left\{u_{\alpha_i}, u_{\alpha_j}\right\}, u_{\alpha_0}, \ldots, \hat{u}_{\alpha_i}, \ldots, \hat{u}_{\alpha_j}, \ldots, u_{\alpha_n}\right) \qquad \text{(B.3)}$$

where $\epsilon_i = [\alpha_i]\big(\sum_{k<i}[\alpha_k]\big)$ and $\epsilon_{ij} = \epsilon_i + \epsilon_j + [\alpha_i][\alpha_j]$.

It obeys $\delta^2 = 0$, so that one can define n-cocycles, which are closed n-cochains ($\delta\chi^{(n)} = 0$), and coboundaries, which are exact n-cochains ($\chi^{(n)} = \delta\chi^{(n-1)}$). As usual, one considers closed cochains modulo exact ones to study the cohomology associated with δ.

Here, we will be mainly concerned with the action of the Chevalley derivation on 2-cochains,

$$(\delta\chi)(u, v, w) = \{u, \chi(v, w)\} - (-1)^{[u][v]}\{v, \chi(u, w)\} + (-1)^{[w]([u]+[v])}\{w, \chi(u, v)\}$$
$$- \chi(\{u, v\}, w) + (-1)^{[v][w]}\chi(\{u, w\}, v) - (-1)^{[u]([v]+[w])}\chi(\{v, w\}, u).$$

We now consider a deformation of the enveloping algebra $\mathcal{U}(\mathcal{A})$,

$$\{u_\alpha, u_\beta\}_\hbar = f_{\alpha\beta}{}^{\gamma}u_\gamma + \hbar\varphi_\hbar(u_\alpha, u_\beta) \qquad \text{(B.4)}$$

where $\varphi_\hbar$ is a 2-cochain which may depend on positive powers of $\hbar$. Asking the bracket $\{\cdot,\cdot\}_\hbar$ to obey the graded Jacobi identity is equivalent to saying that $\varphi_\hbar$ is a 2-cocycle:

$$\delta\varphi_\hbar(u_\alpha, u_\beta, u_\gamma) = 0 \qquad \text{(B.5)}$$

We now prove a result that is used in the present paper.

Lemma B.1. *Let $gl(M|N)_p$ be the polynomial algebra based on $gl(M|N)$, truncated at order p, and u_j^{ab} ($j < p$ and $a, b = 1, \ldots, M+N$) the corresponding generators. Let φ be a 2-cocycle with values in $\mathcal{U}(gl(M|N)_p)$. We introduce $u_j^0 = \sum_{a=1}^{M+N} u_j^{aa}$.*

If $\varphi\big(u_0^{ab}, u_j^{cd}\big)$ and $\varphi\big(u_1^{ab}, u_j^{cd}\big)$, $\forall a, b, c, d = 1, \ldots, M+N$ and $\forall j = 0, \ldots, p-1$ are known, then φ is completely determined up to $\varphi\big(u_j^0, u_k^0\big)$, $j, k > 1$, which is central in $\mathcal{U}(gl(M|N)_p)$.

Proof. We write the cocycle condition for a triplet $\big(u_j^{ab}, u_k^{cd}, u_\ell^{eg}\big)$,

$$\varphi\left(\left\{u_j^{ab}, u_k^{cd}\right\}, u_\ell^{eg}\right) + (-1)^{([c]+[d])([e]+[g])}\varphi\left(\left\{u_j^{ab}, u_\ell^{eg}\right\}, u_k^{cd}\right)$$
$$- (-1)^{([a]+[b])([c]+[d]+[e]+[g])}\varphi\left(\left\{u_k^{cd}, u_\ell^{eg}\right\}, u_j^{ab}\right) = \left\{u_j^{ab}, \varphi\left(u_k^{cd}, u_\ell^{eg}\right)\right\}$$
$$- (-1)^{([a]+[b])([c]+[d])}\left\{u_k^{cd}, \varphi\left(u_j^{ab}, u_\ell^{eg}\right)\right\}$$
$$- (-1)^{([a]+[b])([c]+[d]+[e]+[g])}\left\{u_\ell^{eg}, \varphi\left(u_j^{ab}, u_k^{cd}\right)\right\}. \qquad \text{(B.6)}$$

We write the commutation relations of $gl(M|N)_p$ as

$$\left\{u_j^{ab}, u_k^{cd}\right\} = \delta^{bc}u_{j+k}^{ad} - (-1)^{([a]+[b])([c]+[d])}\delta^{ad}u_{j+k}^{cb} \qquad \text{with} \quad u_n^{ab} = 0 \quad n > p \quad \forall a, b. \qquad \text{(B.7)}$$

Taking as a special case $e = g = a \neq b$ and $\ell = 1$, one obtains from (B.6)

$$\varphi\left(u_{j+1}^{ab}, u_k^{cd}\right) = \left\{u_j^{ab}, \varphi\left(u_k^{cd}, u_1^{aa}\right)\right\} - \varphi\left(\left\{u_j^{ab}, u_k^{cd}\right\}, u_1^{aa}\right)$$
$$+ (-1)^{([c]+[d])([a]+[b])}\left((\delta^{da} - \delta^{ca})\varphi\left(u_{k+1}^{cd}, u_j^{ab}\right)\right.$$
$$\left.- \left\{u_1^{aa}, \varphi\left(u_j^{ab}, u_k^{cd}\right)\right\} - \left\{u_k^{cd}, \varphi\left(u_j^{ab}, u_1^{aa}\right)\right\}\right). \qquad \text{(B.8)}$$

Taking as a special case $j = 1, k = 2$, this last equation shows that one can compute $\varphi\big(u_2^{ab}, u_2^{cd}\big)$, for $a \neq b$, as soon as one knows $\varphi\big(u_1^{cd}, u_j^{eg}\big)$, $\forall c, d, e, g, \forall j$. Then, in the same way, $j = 1$ allows one to compute $\varphi\big(u_2^{ab}, u_{k+1}^{cd}\big)$ as soon as one knows $\varphi\big(u_2^{ab}, u_k^{cd}\big)$.

More generally, if one supposes by induction that $\varphi\big(u_{j'}^{ab}, u_k^{cd}\big)$, $\forall j' \leqslant j$, $\forall k$, and $\forall c, d$, $a \neq b$, are known, (B.8) shows that one can compute $\varphi\big(u_{j+1}^{ab}, u_k^{cd}\big)$, for $a \neq b$ and $\forall k$.

Thus, by induction, we have shown that one can compute $\varphi\big(u_j^{ab}, u_k^{cd}\big)$, for $a \neq b, \forall j, k, c, d$, from the knowledge of $\varphi\big(u_1^{cd}, u_k^{eg}\big)$.

It remains to compute $\varphi\big(u_j^{aa}, u_k^{bb}\big)$. For such a purpose, we start again with (B.6) now with $a = d \neq b = c$ and $e = g$:

$$\begin{aligned}\varphi\left(u_{j+k}^{aa} - (-1)^{[a]+[b]}u_{j+k}^{bb}, u_\ell^{ee}\right) &= (\delta^{ae} - \delta^{be})\left(\varphi\left(u_{j+\ell}^{ab}, u_k^{ba}\right) + (-1)^{[a]+[b]}\varphi\left(u_{k+\ell}^{ba}, u_j^{ab}\right)\right)\\ &\quad + \left\{u_j^{ab}, \varphi\left(u_k^{ba}, u_\ell^{ee}\right)\right\} - (-1)^{[a]+[b]}\left\{u_k^{ba}, \varphi\left(u_j^{ab}, u_\ell^{ee}\right)\right\}\\ &\quad - (-1)^{[a]+[b]}\left\{u_\ell^{ee}, \varphi\left(u_j^{ab}, u_k^{ba}\right)\right\}. \qquad \text{(B.9)}\end{aligned}$$

All the terms in the rhs of the above equation are known, so that one can compute[4] $\varphi\big((-1)^{[a]}u_j^{aa} - (-1)^{[b]}u_j^{bb}, u_k^{ee}\big)$, $\forall a, b, e, \forall j, k$.

Thus, only $\varphi\big(u_j^0, u_k^0\big)$, where $u_j^0 = \sum_{a=1}^{M+N} u_j^{aa}$, remains to be computed. Once again, from (B.6), taking $a = b$ and $c = d$, and then summing over a and d, one obtains

$$\left\{u_\ell^{eg}, \varphi\left(u_j^0, u_k^0\right)\right\} = 0 \qquad \text{(B.10)}$$

which shows that $\varphi\big(u_j^0, u_k^0\big)$ is central in $\mathcal{U}(gl(M|N)_p)$.

Thus, apart from the values $\varphi\big(u_0^{ab}, u_k^{cd}\big)$ and the just mentioned central terms, we are able to compute all the expressions $\varphi\big(u_j^{ab}, u_k^{cd}\big)$. This ends the proof. □

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